ECE 874 Test 1 Spring 2010 **Name** Solution

1. (25pts) Find the equilibrium point for the system and use a Lyapunov analysis to show that it is globally asymptotically stable:

$$\dot{x}_1 = -x_1 - x_2 + c$$
$$\dot{x}_2 = -x_2 + x_1$$

Eq pt: $0 = -x_1 - x_2 + c$

$0 = -x_2 + x_1 \Rightarrow x_1 = x_2$ sub $0 = -2x_2 + c \Rightarrow x_2 = \frac{c}{2}$

$\Rightarrow x_1 = \frac{c}{2}$

eq pt $x = \begin{bmatrix} c/2 \\ c/2 \end{bmatrix}$

change of variables to shift eq pt to $(0,0)$

$\boxed{z_1 = x_1 - \frac{c}{2}} \Rightarrow x_1 = z_1 + \frac{c}{2}$ $\boxed{z_2 = x_2 - c/2} \Rightarrow x_2 = z_2 + \frac{c}{2}$

$\dot{z}_1 = \dot{x}_1$ $\dot{z}_2 = \dot{x}_2$

sub & write new system

$$\dot{z}_1 = -(z_1 + c/2) - (z_2 + c/2) + c = -z_1 - z_2$$
$$\dot{z}_2 = -(z_2 + c/2) + (z_1 + c/2) = -z_2 + z_1$$

Propose $V(x) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$ PD, radially unbounded, $V(0) = 0$

$$\dot{V} = z_1 \dot{z}_1 + z_2 \dot{z}_2$$
$$= -z_1^2 - \cancel{z_1 z_2} - z_2^2 + \cancel{z_1 z_2} = -z_1^2 - z_2^2 \quad \text{ND}$$

Thm 3.2 $\rightarrow$ $z_1 \to 0$ as $t \to \infty$ (GAS)
$z_2 \to 0$ as $t \to \infty$

Thus

$$x_1 = z_1 + c/2 \to c/2 \text{ as } t \to \infty$$
$$x_2 = z_2 + c/2 \to c/2 \text{ as } t \to \infty$$

stability of the

at the origin $x=(0,0)$

2. (20pts) What (if anything) can the $V(x)$ shown below tell about the system?

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(x_1 + x_2) - \sin(x_1 + x_2)$$
$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}(x_1 + x_2)^2$$

$V(0) = 0$, PD, radially unbounded

$$\dot{V} = x_1\dot{x}_1 + (x_1 + x_2)(\dot{x}_1 + \dot{x}_2)$$

$$= x_1 x_2 + (x_1 + x_2)(x_2) + (x_1 + x_2)(-(x_1+x_2) - \sin(x_1+x_2))$$

$$= \cancel{x_1 x_2} + \cancel{x_1 x_2} + \cancel{x_2^2} - x_1^2 - \cancel{x_1 x_2} - x_1 \sin(x_1+x_2) - \cancel{x_1 x_2} - \cancel{x_2^2}$$

$$- x_2 \sin(x_1 + x_2)$$

$$= -x_1^2 - (x_1 + x_2)\sin(x_1 + x_2)$$

$$\leq -(x_1 + x_2)\sin(x_1 + x_2)$$

$$\leq -(x_1 + x_2)(x_1 + x_2) \quad \text{for } |x_1 + x_2| \leq \pi$$

$$\leq -(x_1 + x_2)^2$$

$\dot{V}(x)$ is NSD

Thm 3.1 the eq pt is Locally Stable

$(t \geq 0)$

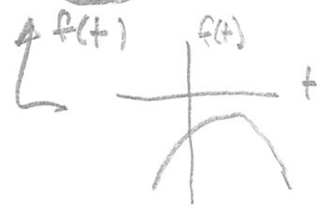
3. (25pts) Analyze the stability of the origin x=0 for the system using W(x,t):

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -(x_1 + x_2)$$
$$W(x,t) = (x_1^2 + x_2^2)(t^2 + 1)$$

$X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point

$x_1^2 + x_2^2 \leq W(x,t)$ not decresent, radially unbounded

$$\dot{W} = (2x_1\dot{x}_1 + 2x_2\dot{x}_2)(t^2+1) + (x_1^2+x_2^2)(2t)$$
$$= \left(2x_1(-x_1+\cancel{x_2}) + 2x_2(-\cancel{x_1}-x_2)\right)(t^2+1) + (x_1^2+x_2^2)(2t)$$
$$= (-2x_1^2 - 2x_2^2)(t^2+1) + (x_1^2+x_2^2)(2t)$$
$$= \left[-2(t^2+1) + 2t\right]\left[x_1^2 + x_2^2\right]$$
$$= \left[\underbrace{-2t^2+2t-2}_{f(t)}\right]\left[x_1^2 + x_2^2\right]$$



local minimum

$-4t + 2 = 0$

$t = \frac{2}{4} = \frac{1}{2}$

$f(t) = -2\left(\frac{1}{4}\right) + 1 - 2 = -\frac{1}{2} - 1$

$= -3/2$

$\therefore f(t) \leq -3/2$

& $f(t) \neq 0$ for any $t \geq 0$

$\leq -3/2\left[x_1^2 + x_2^2\right] \Rightarrow \dot{W}(x,t)$ is negative definite in some domain of the origin

$\Rightarrow$ system is stable at the origin.

4. (5pts) Covert the system into state-space form

$$m\ddot{y} + a\dot{y} + ky + 2y^3 + 2z^3 = f(t)$$

$$\dot{z} = y + a\dot{y} + z$$

$$\left.\begin{aligned} x_1 &= y \\ x_2 = \dot{x}_1 &= \dot{y} \\ \dot{x}_2 &= \ddot{y} \\ x_3 &= z \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left(-a x_2 - k x_1 - 2x_1^3 - 2x_3^3 + f(t)\right) \\ \dot{x}_3 &= x_1 + a x_2 + x_3 \end{aligned}$$

(5 pts) Are the following positive definite, positive semi-definite, negative definite, negative semi-definite, none of the above

$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2$	PD
$V(x) = \frac{1}{2}(x_1 - x_2)^2$	NSD
$V(x) = x^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x$	PD
$V(x) = \frac{1}{2}(x_1)^2 + \frac{1}{2}(x_1)^4 + \frac{1}{2}(x_1)^6$	PSD
$V(x) = \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \cancel{+2}$	PSD

positive integers

equilibrium point at the

5 **(20pts) Find p and q so that $V(x)$ can be used to show that the origin is globally asymptotically stable.**

$$\dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4$$

$$\dot{x}_2 = -x_1 - x_2 + x_1 x_2$$

$$V(x) = x_1^p + x_2^q$$

$$\dot{V} = p x_1^{p-1} \dot{x}_1 + q x_2^{q-1} \dot{x}_2$$

$$= p x_1^{p-1} \left[-x_1 + 2x_2^3 - 2x_2^4 \right]$$

$$+ q x_2^{q-1} \left[-x_2 \underbrace{- x_1 + x_1 x_2}_{} \right]$$

ⓐ

observation 1: would like the terms with both x_1 & x_2 to cancel

a) want

$$0 = -2p x_1^{p-1} x_2^4 + q x_2^{\cancel{q-1}\, q} x_1 \cancel{x_2}$$

$$= -2p x_1^{p-1} x_2^4 + q x_2^q x_1$$

this suggests (but does not guarantee)

$q = 4 \quad p = 2$

observation 2: p & q should be even numbers so that these terms will stabilize the system

try $p = 2, \ q = 4$

$V(x) = x_1^2 + x_2^4$ $\qquad$ $V(0) = 0$, radially unbounded, PD

$$\dot{V} = 2x_1 \dot{x}_1 + 4x_2^3 \dot{x}_2$$

$$= 2x_1 \left(-x_1 + 2x_2^3 - 2x_2^4 \right) + 4x_2^3 \left(-x_1 - x_2 + x_1 x_2 \right)$$

$$= -2x_1^2 + \cancel{4x_1 x_2^3} - \cancel{4x_1 x_2^4} - \cancel{4x_2^3 x_1} - 4x_2^4 + \cancel{4x_2^4 x_1}$$

$$= -2x_1^2 - 4x_2^2 \Rightarrow \dot{V} \text{ is ND}$$

$\therefore \quad x = (0,0)$ is GAS

ECE 874 Test 2 Spring 2010 **Name** Answer

1. **(20pts) Design a tracking controller, *u(t)*, for the system:**

$$\ddot{x} = a\dot{x} + bx + c + u$$

Assume that the desired trajectory, x_d, and the first two derivatives exist and are bounded. Prove that the controller will work and that all signals remain bounded. Use the Lyapunov function candidate $V = \frac{1}{2}e^2 + \frac{1}{2}r^2$ where $e = x_d - x$ and $r = \dot{e} + \alpha e$.

$\dot{V} = e\dot{e} + r\dot{r}$

$= er - \alpha e^2 + r\dot{r}$

$\dot{r} = \ddot{e} + \alpha\dot{e}$

$= \ddot{x}_d - \ddot{x} + \alpha\dot{e}$

$= \ddot{x}_d - \alpha\dot{e} - a\dot{x} - bx - c - u$

$= -\alpha e^2 + er + r\left[\underbrace{\ddot{x}_d - \alpha\dot{e} - a\dot{x} - bx + c}\; - u\right]$

let $u = \ddot{x}_d - \alpha\dot{e} - a\dot{x} - bx - c + r + e$

$\dot{V} = -\alpha e^2 + \cancel{er} - r^2 - er$

$= -\alpha e^2 - r^2$

$\left.\begin{array}{l} V \text{ PD \& radially unbounded} \\ \dot{V} \text{ ND} \end{array}\right\} \Rightarrow e, r \to 0 \Rightarrow \dot{e} \to 0$

$e, \to 0$, x_d bounded $\to x$ is bounded, $x \to x_d$

$\dot{e}, \to 0$, $\dot{x}_d$ bounded $\to \dot{x}$ is bounded, $\dot{x} \to \dot{x}_d$

$u = f_1(\ddot{x}_d, \dot{x}_d, \dot{x}, x)$ is bounded

$\ddot{x} = f_2(\dot{x}, x, u)$ is bounded

2. (20pts) Use a backstepping approach (by defining an intermediate tracking error) to design an adaptive tracking controller for the system:

$$\dot{x}_1 = ax_1 + x_2$$
$$\dot{x}_2 = u$$

where a is an unknown constant parameter. Assume that the desired trajectory and the first two derivatives exist and are bounded. Prove that the controller will work and that all signals remain bounded.
Hint: Use two estimation errors $\tilde{a}_1 = a - \hat{a}_1$ and $\tilde{a}_2 = a - \hat{a}_2$ with a Lyapunov function $V = (e_1^2 + \tilde{a}_1^2 + e_2^2 + \tilde{a}_2^2)\frac{1}{2}$ where $e_1 = x_{1d} - x_1$ and $e_2 = x_{2d} - x_2$.

$$\dot{e}_1 = \dot{x}_{1d} - \dot{x}_1 = \dot{x}_{1d} - ax_1 \underbrace{- x_2 + x_{2d}}_{e_2} - x_{2d}$$
$$= \dot{x}_{1d} - ax_1 - x_{2d} + e_2$$

$\boxed{* \text{ let } x_{2d} = \dot{x}_{1d} - \hat{a}_1 x_1 - e_1} \Rightarrow \dot{e}_1 = -\tilde{a}_1 x_1 - e_1 + e_2$

$$V_1 = (e_1^2 + \tilde{a}_1^2)\tfrac{1}{2}$$
$$\dot{V}_1 = e_1\dot{e}_1 - \tilde{a}_1\dot{\hat{a}}_1$$
$$= e_1(-\tilde{a}_1 x_1 - e_1 + e_2) - \tilde{a}_1\dot{\hat{a}}_1$$
$$= -e_1^2 + e_1 e_2 + \tilde{a}_1(-e_1 x_1 - \dot{\hat{a}}_1)$$

$\boxed{* \text{ let } \dot{\hat{a}}_1 = -e_1 x_1 \rightarrow \hat{a}_1 = -\int_0^t e_1 x_1 \, dt}$

$$\dot{V}_1 = -e_1^2 + e_1 e_2$$

derivative

$$\dot{e}_2 = \dot{x}_{2d} - \dot{x}_2 = \ddot{x}_{1d} - \dot{\hat{a}}_1 x_1 - \hat{a}_1\dot{x}_1 - \dot{e}_1 - u$$
$$= \ddot{x}_{1d} + (e_1 x_1)x_1 - \hat{a}_1\dot{x}_1 - \dot{x}_{1d} + \dot{x}_1 - u$$
$$= \ddot{x}_{1d} + e_1 x_1^2 - \dot{x}_{1d} + (1 - \hat{a}_1)\dot{x}_1 - u$$

system $\dot{x}_1 = ax_1 + x_2$

$$= \underbrace{\ddot{x}_{1d} + e_1 x_1^2 - \dot{x}_{1d} + (1 - \hat{a}_1)(x_2)}_{\text{cancel with } u} + (1 - \hat{a}_1)ax_1 - u$$

cancel

let $u = \ddot{x}_{1d} + e_1 x_1^2 - \dot{x}_{1d} + (1 - \hat{a}_1)x_2 + (1 - \hat{a}_1)\hat{a}_2 x_1 - e_2 - e_1$

$$\dot{e}_2 = (1 - \hat{a}_1)\tilde{a}_2 x_1 - e_2 - e_1$$

$$\dot V = \dot V_1 + e_2 \dot e_2 - \tilde a_2 \dot{\hat a}_2$$

$$= -e_1^2 + \cancel{e_1 e_2} + e_2(1-\hat a_1)\tilde a_2 x_1 - e_2^2 - \cancel{e_1 e_2} - \tilde a_2 \dot{\hat a}_2$$

$$= -e_1^2 - e_2^2 + \left[e_2(1-\hat a_1)x_1 - \dot{\hat a}_2\right]\tilde a_2$$

$$\dot{\hat a}_2 = x_1(1-\hat a_1)e_2 \Rightarrow \hat a_2 = \int_0^t x_1(1-\hat a_1)e_2\,dt$$

$\dot V = -e_1^2 - e_2^2$ $\Rightarrow$ e_1 & e_2 are bounded

$$g(x) = -e_1^2 - e_2^2$$

$$\dot g(x) = -e_1 \dot e - e_2 \dot e_2$$

3. (20pts) Design an observer for $\dot{x}$ in the open-loop system

$$\ddot{x} = a\sin(x) + bx + u$$

Prove the performance of your design.

$$\tilde{x} = x - \hat{x}$$

$$\ddot{\tilde{x}} = \ddot{x} - \ddot{\hat{x}} = a\sin(x) + bx - \ddot{\hat{x}}$$

$$s = \dot{\tilde{x}} + \alpha\tilde{x}$$

$$\dot{s} = \ddot{\tilde{x}} + \alpha\dot{\tilde{x}} = a\sin(x) + bx - \ddot{\hat{x}} + \alpha\dot{\tilde{x}}$$

$$V = \tfrac{1}{2}\tilde{x}^2 + \tfrac{1}{2}s^2$$

$$\dot{V} = \tilde{x}\dot{\tilde{x}} + s\dot{s} = \tilde{x}(s - \alpha\tilde{x}) + s\dot{s}$$

$$= -\alpha\tilde{x}^2 + \tilde{x}s + s(a\cancel{\sin(x)} + \cancel{bx} - \ddot{\hat{x}} + \cancel{\alpha\dot{\tilde{x}}})$$

$$\ddot{\hat{x}} = a\cancel{\sin(x)} + \cancel{bx} + \cancel{\alpha\dot{\tilde{x}}} + \tilde{x} + s$$

$$\dot{V} = -\alpha\tilde{x}^2 + \cancel{\tilde{x}s} - s^2 - \cancel{\tilde{x}s}$$

$$= -\alpha\tilde{x}^2 - s^2$$

V PD, $\dot{V}$ ND $\rightarrow$ $\tilde{x}, s \rightarrow 0 \Rightarrow \dot{\tilde{x}} \rightarrow 0$

put $\ddot{\hat{x}}$ into an implementable form:

$$\dot{\hat{x}} = p + (\text{stuff to differentiate in } \ddot{\hat{x}})$$

$$\dot{p} = (\text{stuff not to differentiate in } \ddot{\hat{x}})$$

$$\ddot{\hat{x}} = a\sin(x) + bx + \alpha\dot{\tilde{x}} + \tilde{x} + \dot{\tilde{x}} + \alpha\tilde{x}$$

$$= \underbrace{a\sin(x) + bx + (1+\alpha)\tilde{x}}_{\text{put in } p} + \underbrace{(1+\alpha)\dot{\tilde{x}}}_{\text{put integral in } \dot{\hat{x}}}$$

Final Form

$$\left[\begin{array}{l} \dot{\hat{x}} = p + (1+\alpha)\tilde{x} \\ \dot{p} = a\sin(x) + bx + (1+\alpha)\tilde{x} \end{array}\right.$$

4. **(10pts) Describe a control design approach suitable for regulation of each of the following systems**

$\dot{x} = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 - x_2)^2 + u$	exact model knowledge, cancel nonlinearities and add stabilizing term
$\dot{x} = \sin(ax) + u$ where a is an unknown constant parameter	not linear in parameters (can't use adaptive), use robust to compensate for $\sin(ax)$ term
$\dot{x} = b\sin(x) + u$ where a is an unknown constant parameter	adaptive control
$\ddot{x} = b\sin(x) + u$ b is known constant parameter $\dot{x}$ is not measurable	observer for $\dot{x}$, co-design observer and controller
$\ddot{x} = f(x) + u$ where $f(x)$ is an unknown function of only x	robust control

5. **(10pts) You have come to the point in your analysis where you find:**

$$\dot{V} = -k_1 s^2 - xs - k_2 x^2,$$

finish the analysis by showing $\dot{V}$ is negative semi-definite in either x or s (your choice). What are the conditions on k_1 and k_2?

$$\dot{V} \leq -k_1 s^2 + |x||s| - k_2 |x|^2$$

$$\leq -k_1 s^2 + |x| \underbrace{(|s| - k_2|x|)}$$

case 1: $|s| > k_2|x|$ then $< |s|$

2: $|s| < k_2|x|$ then < 0

$|s|$ is largest upper bound

$$\Rightarrow \frac{|s|}{k_2} < |x|$$

$$\leq -k_1 s^2 + \frac{|s|}{k_2}|s|$$

$$\leq -s^2\left(k_1 - \frac{1}{k_2}\right) \qquad k_1 > \frac{1}{k_2}$$

6. **(20pts) Design a singularity free tracking controller, *u*, for the system:**

$$\left(x^2+1\right)\ddot{x} = -x+u$$

Assume that the desired trajectory and the first two derivatives exist and are bounded. Prove that the controller will work and that all signals remain bounded.

$r = \dot{e} + \alpha e$

$\dot{r} = \ddot{e} + \alpha \dot{e} = \ddot{x}_d - \ddot{x} + \alpha \dot{e}$

$(x^2+1)\dot{r} = (x^2+1)\ddot{x}_d + x - u + (x^2+1)\alpha \dot{e}$

$V = \frac{1}{2}(x^2+1)r^2$

$\dot{V} = x\dot{x}r^2 + (x^2+1)r\dot{r}$

$= x\dot{x}r^2 + r\left[(x^2+1)\ddot{x}_d + x + (x^2+1)\alpha\dot{e} - u\right]$

let $u = (x^2+1)\ddot{x}_d + x + (x^2+1)\alpha\dot{e} + r + x\dot{x}r$

$\dot{V} = -r^2$

V PD, $\dot{V}$ ND $\rightarrow r \rightarrow 0 \Rightarrow e = \frac{r}{s+1} \rightarrow 0 \Rightarrow \dot{e} \rightarrow 0$

$x_d, \dot{x}_d, \ddot{x}_d$ bounded $\rightarrow u$ is bounded

ECE 874 Final Exam Spring 2010 Name ANS.

1. **(25pts)** Design a robust, sliding mode, controller, u(t), for the system:

x & $\dot{x}$ are measureable

$$\ddot{x} = a\sin(\dot{x}) + b\cos(x+\pi/2) + 2 + u$$

where a and b are unknown constants between 1 and 5, i.e. $1<a<5$ and $1<b<5$. Assume that the desired trajectory, x_d, and the first two derivatives exist and are bounded.. Prove your proposed controller achieves exponential tracking using the Lyapunov function candidate $V = \frac{1}{2}e^2 + \frac{1}{2}r^2$ where $e = x_d - x$ and $r = \dot{e} + \alpha e$. Show all signals are bounded.

$$\dot{r} = \ddot{e} + \alpha\dot{e} = \ddot{x}_d - \ddot{x} + \alpha\dot{e} = \ddot{x}_d - a\sin(\dot{x}) - b\cos(x+\pi/2) - 2 - u + \alpha\dot{e}$$

$$V = \tfrac{1}{2}e^2 + \tfrac{1}{2}r^2$$

$$\dot{V} = e\dot{e} + r\dot{r}$$

$$= e(r - \alpha e) + r(\ddot{x}_d - a\sin(\dot{x}) - b\cos(x+\pi/2) - 2 - u + \alpha\dot{e})$$

design $u = \ddot{x}_d - 2 + e + r + u_r + \alpha\dot{e}$

u_r ← term to be designed

$$= -\alpha e^2 - r^2 + r(\underbrace{-a\sin(\dot{x}) - b\cos(x+\pi/2)}_{f(x)} - u_r)$$

$f(x)$, note that $|f(x)| < 5\cdot 1 + 5\cdot 1 = 10$

design $\rho(x) = 10$

design $u_r = 10\dfrac{r}{|r|}$

Note: The constant term "2" could have been included in $\rho(x)$

$$\dot{V} \le -\alpha e^2 - r^2 + 10|r| - r\left(\frac{10r}{|r|}\right)$$

$$\dot{V} \le -\alpha e^2 - r^2 + 10|r| - 10\frac{r^2}{|r|}$$

$$\dot{V} \le -\alpha e^2 - r^2 + 10|r| - 10|r|$$

$$\dot{V} \le -\alpha e^2 - r^2$$

V is PD, $\dot{V}$ is ND $\longrightarrow$ $e, r \to 0$ exponentially fast

$r, e \to 0 \Rightarrow \dot{e} = r - \alpha e \to 0$

x_d bounded, $e \to 0 \Rightarrow x \to x_d$ and is bounded

$\dot{x}_d$ bounded, $\dot{e} \to 0 \Rightarrow \dot{x} \to \dot{x}_d$ and is bounded

$u_r = \text{sgn}(r)$ is bounded $\forall r$

$\ddot{x}_d$ bounded, $e \to 0$, $r \to 0$, u_r bounded $\Rightarrow u$ is bounded

$\dot{x}, x, u$ bounded $\Rightarrow \ddot{x}$ is bounded

2 (cont.)

design $D = \frac{L}{V_b}\left(-f(\cdot) - z + \frac{1}{c}e + u_r\right)$

$$\dot{V} = -e^2 - z^2 + \frac{1}{c} e z \dot{I}_{pv} - u_r z$$

$$\dot{V} \leq -e^2 - z^2 + \frac{1}{c}|e||z||\dot{I}_{pv}| - u_r z$$

design $u_r = \frac{1}{c}|e| \mu \frac{z}{|z|}$

$$\dot{V} \leq -e^2 - z^2 + \frac{1}{c}|e|\mu\left(\cancel{|z|} - \cancel{|z|}\right)$$

$$\dot{V} \leq -e^2 - z^2$$

V PD, $\dot{V}$ ND $\rightarrow$ $e, z \rightarrow 0$

V_d bounded, $e \rightarrow 0$ $\Rightarrow$ V_{pv} is bounded

V_{pv} bounded, assumption 4 $\Rightarrow$ I_{pv} is bounded

$I_d = g(e, \dot{V}_d, I_{pv})$ is bounded

z, I_d bounded $\Rightarrow$ I_L is bounded

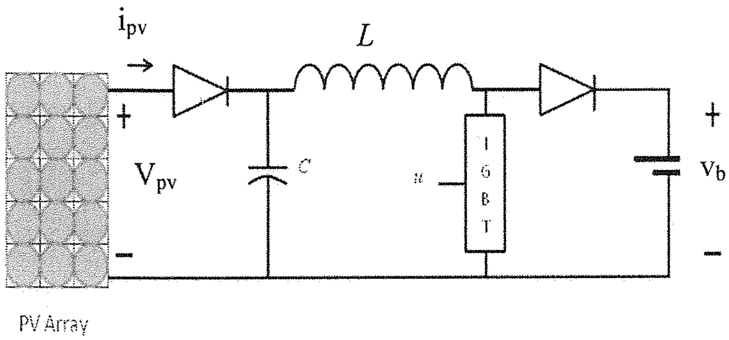
I_L, I_{pv} bounded $\Rightarrow$ $\dot{V}_{pv}$ is bounded

e, μ, z bounded $\Rightarrow$ u_r bounded

$D = g_1(f(\cdot), z, e, u_r)$ $\Rightarrow$ D is bounded

V_{pv}, D bounded $\Rightarrow$ $\dot{I}_L$ is bounded

2. (25pts) A photovoltaic battery charging circuit is shown by the circuit below.



Assumption 1: $V_{pv}(t)$, $I_{pv}(t)$, $I_L(t)$ and $V_b(t)$ are measurable.

Assumption 2: C and L are known constants.

Assumption 3: V_b is modeled as a constant value due to its slow charge dynamics [5].

Assumption 4: $I_{pv}(t)$ is bounded provided that $V_{pv}(t)$ is bounded.

Assumption 5: $\dot{I}_{pv}(t)$ can be upper bounded by a positive constant such that $|\dot{I}_{pv}| < \mu$ where $\mu \in \mathbb{R}^+$.

The system can be transformed from discrete commands for u to the IGBT into the following continuous differential equations where D represents the duty cycle to the IGBT and is the control input:

$$C\dot{V}_{pv} = I_{pv} - I_L$$
$$L\dot{I}_L = V_{pv} - DV_b$$

Desired trajectory
Assume $V_d, \dot{V}_d, \ddot{V}_d$ exist and are bounded

Using integrator backstepping and the definitions $e = V_d - V_{pv}$ and $z = I_d - I_L$ to design a controller D. You must explicitly differentiate I_d and show all signals are bounded for full credit.

$$\dot{e} = \dot{V}_d - \dot{V}_{pv} = \dot{V}_d - \frac{1}{C}(I_{pv} - I_L) + \frac{1}{C}I_d - \frac{1}{C}I_d$$
$$= \dot{V}_d - \frac{1}{C}I_{pv} + \frac{1}{C}I_L - \frac{1}{C}I_d + \frac{1}{C}I_d$$
$$= \dot{V}_d - \frac{1}{C}I_{pv} + \frac{1}{C}I_d - \frac{1}{C}z$$

$$V_1 = \tfrac{1}{2}e^2$$
$$\dot{V}_1 = e\dot{e} = e\left(\cancel{\dot{V}_d - \frac{1}{C}I_{pv}} + \frac{1}{C}I_d - \frac{1}{C}z\right)$$

design $I_d = C\left(-e - \dot{V}_d + \frac{1}{C}I_{pv}\right)$

$$= -e^2 - \frac{1}{C}ez$$

$$\dot{z} = \dot{I}_d - \dot{I}_L = C\left(-\dot{e} - \ddot{V}_d + \frac{1}{C}\dot{I}_{pv}\right) - \frac{1}{L}(V_{pv} - DV_b)$$
$$= -C\left[\dot{V}_d - \frac{1}{C}(I_{pv} - I_L)\right] - C(\ddot{V}_d) + \frac{1}{C}C\dot{I}_{pv} - \frac{1}{L}V_{pv} + \frac{1}{L}V_b D$$

$$V = V_1 + \tfrac{1}{2}z^2$$
$$\dot{V} = \dot{V}_1 + z\dot{z}$$
$$= -e^2 - \frac{1}{C}ez + z\underbrace{\left(-C\left[\dot{V}_d - \frac{1}{C}(I_{pv} - I_L) - \ddot{V}_d\right] - \frac{1}{L}V_{pv}\right)}_{f(I_{pv}, I_L, V_d, \dot{V}_d, \ddot{V}_d)\text{ is a measurable function}}$$
$$+ \frac{1}{C}ez\dot{I}_{pv} + \frac{1}{L}V_b D$$

3. **(25 pts) Calculate the control law for u in the magnetic suspension to transform the system into a linear, controllable system. Verify the transformation.**

Consider the schematic representation of a magnetic suspension shown in figure 2. The input is the voltage source. The modeling details a motion are

$$\dot{x} = y$$

$$\dot{y} = g - \frac{\alpha z^2}{M(2x+\beta)^2} + F_d(t)$$

$$\dot{z} = \frac{2x+\beta}{\alpha}(u - Rz) + \frac{2yz}{2x+\beta}$$

where $\alpha = \mu_0 N^2 A$ and $\beta = \frac{L_1}{\mu_1} + \frac{L_2}{\mu_2}$

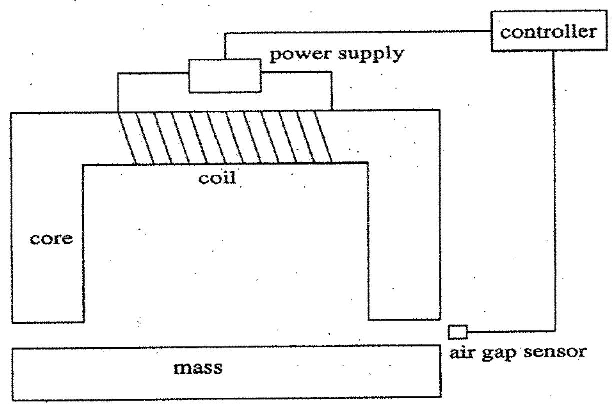


Figure 2

x is the displacement of the isolated mass measured from the magnet pole face, z is the current in the coil, and u is the voltage. $F_d(t)$ is the disturbance force, assume it is a known constant.

The parameters are:

Mass M	8.91 kg
Area of cross section A	0.00025 m^2
Resistance of coil R	3.0 Ω
Path length in core L_1	0.28 m
Path length in mass L_2	0.05 m
Permeability of free space μ_0	1.26 x 10^{-6} Hm^{-1}
Relative permeability of core μ_1	10000
Relative permeability of mass μ_2	100
Number of turns N	800

put into state-space form using $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = x$, $x_2 = y$, $x_3 = z$

$$\dot{\bar{x}} = \underbrace{\begin{bmatrix} x_2 \\ g - \frac{\alpha x_3^2}{M(2x_1+\beta)^2} + F_d(t) \\ -\frac{2x_1+\beta}{\alpha}(Rx_3) + \frac{2x_2x_3}{2x_1+\beta} \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{2x_1+\beta}{\alpha} \end{bmatrix}}_{g(x)} u \qquad z = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix}$$

Constraints

✓ ① $\frac{\partial T_1}{\partial x} g(x) = 0 = \frac{\partial T_1}{\partial x_3}\left(\frac{2x_1+\beta}{\alpha}\right) = 0$ ③ $\frac{\partial T_3}{\partial x} g(x) \neq 0 \neq \frac{\partial T_3}{\partial x_3}\left(\frac{2x_1+\beta}{\alpha}\right)$

✓ ② $\frac{\partial T_2}{\partial x} g(x) = 0 = \frac{\partial T_2}{\partial x_3}\left(\frac{2x_1+\beta}{\alpha}\right) = 0$

$$\frac{\partial T_1}{\partial x} f(x) = T_2$$

④ $$\frac{\partial T_1}{\partial x_1}(x_2) + \cancelto{0 \text{ by } (a)}{\frac{\partial T_1}{\partial x_2}}\left(g - \frac{\alpha x_3^2}{M(2x_1+\beta)} + F_d(t)\right) + \cancelto{0 \text{ by } (a)}{\frac{\partial T_1}{\partial x_3}}\left(-\frac{2x_1+\beta}{\alpha}(Rx_3) + \frac{2x_2 x_3}{2x_1+\beta}\right) = T_2$$

$$\frac{\partial T_2}{\partial x} f(x) = T_3$$

⑤ $$\cancelto{0 \text{ by } (b)}{\frac{\partial T_2}{\partial x_1}}(x_2) + \frac{\partial T_2}{\partial x_2}\left(g - \frac{\alpha x_3^2}{M(2x_1+\beta)} + F_d(t)\right) + \cancelto{0 \text{ by } (b)}{\frac{\partial T_2}{\partial x_3}}\left(\frac{-2x_1+\beta}{\alpha}(Rx_3) + \frac{2x_2 x_3}{2x_1+\beta}\right) = T_3$$

choose $\underset{(a)}{T_1 = x_1} \Rightarrow \frac{\partial T_1}{\partial x_2} = \frac{\partial T_1}{\partial x_3} = 0$ ① is satisfied

from ④ $\frac{\partial T_1}{\partial x_1} x_2 = \underset{(b)}{x_2 = T_2} \Rightarrow \frac{\partial T_2}{\partial x_1} = \frac{\partial T_2}{\partial x_2} = 0$ ② is satisfied

④ is satisfied

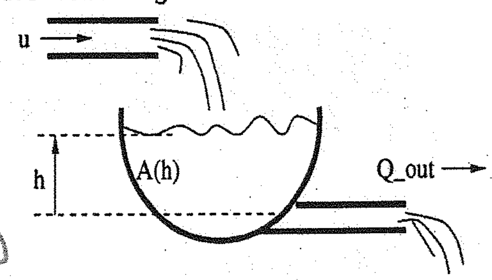
from ⑤

$$\left(g - \frac{\alpha x_3^2}{M(2x_1+\beta)} + F_d(t)\right) = T_3$$

4.

Consider the problem of controlling the height, h, of fluid in the tank shown
With the fluid level starting at some initial height, h_0, the goal is to choose the input flow rate, u, which brings h as close as possible to h_d, the desired height. The tank has a cross-sectional area, A(h), which is a function of the height of the fluid in the tank. A spigot at the bottom of the tank leaks fluid at a rate proportional to the square root of the height of fluid in the tank, $Q_{out} = a\sqrt{2gh}$, where a is a constant of proportionality and g is the acceleration of gravity. Conservation of mass gives the following nonlinear differential equation governing the height of fluid in the tank:

$$A(h)\dot{h} = u - a\sqrt{2gh}$$



≥ 0 (if $h \geq 0$)

a. **(10 pts) Assume $A(h) = h^2+0.1$ and all parameters are exactly known and h can be measured, design a controller so that h tracks h_d. Show all work and that all signals are bounded.**

$A(h) \neq 0$ $\quad \dot{h} = \dfrac{-a\sqrt{2g}\, h^{1/2}}{h^2+0.1} + \dfrac{u}{h^2+0.1}$

let $e = h_d - h$

$V = \frac{1}{2}e^2$

$\dot{V} = e\dot{e} = e(\dot{h}_d - \dot{h}) = e\left(\dot{h}_d + \dfrac{a\sqrt{2g}\, h^{1/2}}{h^2+0.1} - \dfrac{u}{h^2+0.1}\right)$

design $u = (h^2+0.1)\left[\dot{h}_d + \dfrac{a\sqrt{2g}\, h^{1/2}}{h^2+0.1} + e\right]$

$\dot{V} = -e^2$

V is PD, $\dot{V}$ is ND $\Rightarrow e \to 0$

h_d bounded by assumption, $e \to 0 \Rightarrow h \to h_d$ is bounded

from problem & definition of h, assume $h \geq 0$

$\Rightarrow u$ is bounded

b. (15 pts) Assume $A(h) = h^2 + 0.1$ and all parameters are exactly known except "a" and h can be measured, design an adaptive controller so that h tracks h_d. Show all work and that all signals are bounded.

$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{a}^2$ where $\tilde{a} = a - \hat{a}$, $\dot{\tilde{a}} = -\dot{\hat{a}}$

$$\dot{V} = e(\dot{h}_d) + a\left(\frac{e\sqrt{2g}\,h^{1/2}}{h^2+0.1}\right) - \frac{e}{h^2+0.1}u - \tilde{a}\dot{\hat{a}}$$

design $u = (h^2+0.1)\left[\dot{h}_d + \hat{a}\left(\frac{e\sqrt{2g}\,h^{1/2}}{h^2+0.1}\right) + e\right]$

$$\dot{V} = \tilde{a}\left[\frac{e\sqrt{2g}\,h^{1/2}}{h^2+0.1} - \dot{\hat{a}}\right] - e^2$$

design $\dot{\hat{a}} = \dfrac{e\sqrt{2g}\,h^{1/2}}{h^2+0.1}$

$\dot{V} = -e^2$

$V \in \mathcal{L}_\infty \Rightarrow \tilde{a}, e \in \mathcal{L}_\infty$

e, h_d bounded by assumption $\Rightarrow h$ is bounded

$\tilde{a} \in \mathcal{L}_\infty$, a constant $\Rightarrow \hat{a}$ is bounded

$\Rightarrow u$ is bounded

assume $h \geq 0$ & bounded; u bounded $\Rightarrow \dot{h}$ is bounded

$\dot{e} = \dot{h}_d - \dot{h}$; $\dot{h}_d$ bounded by assumption $\Rightarrow \dot{e}$ is bounded

let $g(t) = e^2$ then $\dot{g}(t) = 2e\dot{e}$ is bounded

and $\lim_{t\to\infty} e^2 = 0 \Rightarrow e \to 0$.